Frequency dependent and Nonuniform Parameters Transmission Line Model

Agnieszka Wardzińska
Poznan University of Technology
Faculty of Electronics and Telecommunications
Poznań, Poland
agnieszka.wardzinska@put.poznan.pl

Wojciech Bandurski
Poznan University of Technology
Faculty of Electronics and Telecommunications
Poznań, Poland
wojciech.bandurski@put.poznan.pl

***Abstract*— The paper presents a fast and effective method of modeling a nonuniform and dispersive interconnect with frequency dependent parameters. The model uses the S-parameters. The approach is based on the method of successive approximations. The per-unit-length parameters frequency dependence is taken to account by means of the rational approximation – vector fitting. Then the first order approximation of the scattering parameters is calculated. The first order approximation of the S12, S21-parameters is corrected by using second order approximation. Comparisons of the calculated results with the exact one are performed for the Bessel dispersive transmission line.**

Keywords—Interconnect, VLSI, NUTL, Scattering Parameters, Transmission line, SPICE

I. Introduction

Modeling of transmission lines in the time-domain is an ongoing challenge in the simulation of integrated circuits and/or printed circuit boards at high frequency. Nonuniform Transmission Lines are used in RF and microwave circuits.

There can be find many methods and approaches and it is not possible to discuss all of them. For the nonuniform transmission line there is derived especially many numerical and analytical methods but there can be also find some analytical research for example [1,7]. In [1] the author presents the analytical solution using chain matrix parameters of NTLs to calculate the arbitrary lossy and disperssive NTLs. In [7] nonuniformities are considered as perturbations with respect to a nominal uniform line. The approach yields second-order ordinary distributed differential equations with source terms. Additionally it is worth mentioning two other works. In the first paper [2], the author presents an approach based on dyadic Green's function and vector fitting of per-unit-length impedance and admittance of transmission line to obtain a Z matrix of transmission line as a two-port. The line impedance and admittance are the sums of rational functions of complex frequency s, which facilitates the transformation to the time-domain and modeling in SPICE. The biggest problem is the necessity to take into account a large number of terms in every entry of the mentioned Z matrix. In [3], the same author has extended the above approach to weakly nonuniform transmission lines. In that case the author used results obtained for uniform case and parametric macromodeling to obtain the approximate Z matrix of the line. In both papers, the presented approach has been extended to the case of a multiconductor line. On the other hand in paper [4] was developed a method to convert of differential telegrapher's equations into integral equations and next to solve them using the method of successive approximation. In that approach, we obtain a first order approximation of the solution in a simple analytical form which is valid for low loss transmission lines. The drawback of that approach was not including the skin effect and dielectric dispersion.

Our previous research [6] gives the solution of the first approximation of the scattering parameters both in frequency and time domain. Now we extend this solution and take into account the first few samples calculated for the second approximation of scattering parameter S_{12}. The approach base on the method of successive approximations [4], taking into account the line parameter dependence on the frequency and longitudinal coordinate. For this purpose, as in [2,3], we use the concept of rational approximation of per-unit-length parameters of the line in the frequency domain. Our approach is based on scattering parameters of the transmission line.

The paper is organized as follows. The next section presents the integral equations approach to the dispersive transmission line. In the third section, we employ the method of successive approximation to calculate the scattering parameters of a nonuniform transmission line. In the fourth section we present, the calculations for the Bessel frequency dependent transmission line. We conclude in the last section.

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II. Telegrapher's equations in itegral form

The equations for a nonuniform, dispersive transmission line are the following:

$$-\frac{dV(s,y)}{dz} = Z(s)r(z)\,I(s,z),$$
$$-\frac{dI\,(s,y)}{dz} = Y(s)g(z)V(s,z), \tag{1}$$

where

$$Z(s) = Z_o(s) + Z_1(s), Y(s) = Y_o(s) + Y_1(s),$$
$$Z_o(s) = R + sL, \qquad Y_o(s) = G + sC$$
$$Z_1(s) = \sum_{m=1}^{N_z} \frac{R_m^z}{s + p_m^z}, Y_1(s) = \sum_{m=1}^{N_y} \frac{R_m^y}{s + p_m^y}$$
$$z_1 \le z \le z_2, \qquad d = z_2 - z_1$$

d-length of the line

r(z), g(z)- transmission line taper.

In (1) Z_1 and Y_1 have rational form of per-unit-length impedance and admittance of the transmission line obtained as in [2] by means of the vector fitting technique [5]. The next step is introducing current waves instead of voltage and current into the transmission line equations (1). It is done, similarly as in [4], by transformations:

$$\begin{bmatrix} I_- \\ I_+ \end{bmatrix}(z,s) = \boldsymbol{S}(z)\begin{bmatrix} V \\ I \end{bmatrix}(z,s), \boldsymbol{S} = \frac{1}{2}\begin{bmatrix} \sqrt{Y_c} & -\sqrt{Z_c} \\ \sqrt{Y_c} & \sqrt{Z_c} \end{bmatrix}$$
$$Y_c = \sqrt{\frac{g(z)}{r(z)}} = f_c(z)^{-1}. \quad (2)$$

Using transformation (2) we can pass to (3):

$$-\frac{d}{dz}\begin{bmatrix} I_- \\ I_+ \end{bmatrix} = \left\{\boldsymbol{S}\frac{d\boldsymbol{S^{-1}}}{dz} + \boldsymbol{S}\begin{bmatrix} 0 & Z \\ Y & 0 \end{bmatrix}\boldsymbol{S^{-1}}\right\}\begin{bmatrix} I_- \\ I_+ \end{bmatrix}. \quad (3)$$

Equations (3), after differentiation and simple algebraic operations, take the following scalar form:

$$-\frac{dI_-}{dz} = -\left(Q_1\sqrt{rg}\right)I_- + \left(-Q_2\sqrt{rg} + N\right)I_+,$$
$$-\frac{dI_+}{dz} = \left(Q_2\sqrt{rg} + N\right)I_- + \left(Q_1\sqrt{rg}\right)I_+. \quad (4)$$

where:

$$Q_{1/2}(s) = \tfrac{1}{2}\left(R_oY(s) \pm {R_o}^{-1}Z(s)\right), R_o = {R_o}^{-1} = \sqrt{\tfrac{L}{C}},$$
$$N(z) = \frac{1}{2}\frac{d}{dz}\ln\left(f_c(z)\right).$$

We simplify equations (4) by removing diagonal terms in (4) in the following way:

$$\frac{d}{dz}\left(I_-e^{-Q_1\alpha(z,z_1)}\right) = \left(Q_2\sqrt{rg} - N\right)e^{-Q_1\alpha(z,z_1)}I_+,$$
$$\frac{d}{dz}\left(I_+e^{-Q_1\alpha(z,z_1)}\right) = -\left(Q_2\sqrt{rg} + N\right)e^{-Q_1\alpha(z,z_1)}I_-, \quad (5)$$

where:

$$\alpha(y,x) = \int_x^y \sqrt{r(x)g(x)}\,dx.$$

Integrating the first of equations (5) from z to z_2 and the second one from z_1 to z, after simple but tedious manipulations we obtain the system of integral equations for the nonuniform dispersive transmission line. The appropriate solutions of these integral equations for current waves are presented in [6].

III. Scattering Parameters For Dispersive Nonuniform Transmission Line

A. *Scattering parameters for disspersive nonuniform transmission line*

Basing on the results–presented in [6] the scattering parameters for nonuniform dispersive transmission line have the following form:

$$b_1 = S_{11}a_1 + S_{12}a_2,$$
$$b_2 = S_{21}a_1 + S_{22}a_2, \quad (6)$$

where:

$$a_1 = I_+(z_1,s),\ a_2(s) = I_-(z_2,s),$$
$$b_1 = I_-(z_1,s),\ b_2(s) = I_+(z_2,s),$$
$$S_{11}(s) = \left[-\textstyle\sum_{i=0}^{\infty}\left(\Im^-(z_2,z)\Im^+(z,z_1)\right)^i\Im^-(z_2,z)\,e^{-Q_1\alpha(z,z_1)}\right]_{z=z_1},$$
$$S_{22}(s) = \left[-\textstyle\sum_{i=0}^{\infty}\left(\Im^+(z,z_1)\Im^-(z_2,z)\right)^i\Im^+(z,z_1)\,e^{-Q_1\alpha(z_2,z)}\right]_{z=z_2}, \quad (7)$$
$$S_{12}(s) = S_{21}(s) = \left[\textstyle\sum_{i=0}^{\infty}\left(\Im^-(z_2,z)\Im^+(z,z_1)\right)^i\,e^{-Q_1\alpha(z_2,z)}\right]_{z=z_1},$$

$$\Im^-(z_2,z)\{*\} = \int_z^{z_2} Q_-(\xi)\,e^{-Q_1\alpha(\xi,z)}\{*\}d\xi,$$
$$\Im^+(z,z_1)\{*\} = \int_{z_1}^{z} Q_+(\xi)e^{-Q_1\alpha(z,\xi)}\{*\}d\xi.$$

The scattering parameters have the form of infinite series (7). Each term in these three series (7) is the integral of its predecessor. The integration of successive terms in these series can be done analytically or for more complex nonuniformities numerically.

The convergence of the proposed series calculation are analysed in [6]

B. *First approximation of scattering parameters for the Bessel transmission line*

Let us consider the first order approximation of the series (7). It means that we take terms in (7) for i = 0 only. Then we obtain relationships:

$$S_{11}^0(s) = \left[-\Im^-(z_2,z)e^{-Q_1\alpha(z,z_1)}\right]_{z=z_1} = -\int_{z_1}^{z_2} Q_-(\xi,s)e^{-2Q_1\alpha(\xi,z_1)}\,d\xi, \quad (8a)$$
$$S_{22}^0(s) = \left[-\Im^+(z_1,z)e^{-Q_1\alpha(z_2,z)}\right]_{z=z_2} = -\int_{z_1}^{z_2} Q_+(\xi,s)e^{-2Q_1\alpha(z_2,\xi)}\,d\xi, \quad (8b)$$
$$S_{12}(s) = S_{21}(s) = \left[e^{-Q_1\alpha(z_2,z)}\right]_{z=z_1}. \quad (8c)$$

The PUL parameters of the Bessel transmission line are $Z(s)z^{\alpha}$ and $Y(s)z^{\beta}$. By substituting the above PUL to equations (10) and performing integrations we obtain:

$$So_{11}(s) = \tfrac{Q_2(s)}{2Q_1(s)}\left[e^{-2Q_1(s)q\left({z_2}^{1/q}-{z_1}^{1/q}\right)} - 1\right] + \tfrac{\alpha-\beta}{4}qe^{Q_1(s)q{z_1}^{1/q}}\left\{Ei\left(1,2Q_1(s)q{z_1}^{1/q}\right) - Ei\left(1,2Q_1(s)q{z_{21}}^{1/q}\right)\right\}, \quad (9a)$$

$$So_{22}(s) = \tfrac{Q_2(s)}{2Q_1(s)}\left[e^{-2Q_1(s)q\left({z_2}^{1/q}-{z_1}^{1/q}\right)} - 1\right] - \tfrac{\alpha-\beta}{4}qe^{-Q_1(s)q{z_2}^{1/q}}\left\{Ei\left(1,-2Q_1(s)q{z_1}^{1/q}\right) - Ei\left(1,-2Q_1(s)q{z_2}^{1/q}\right)\right\}, \quad (9b)$$

$$So_{12}(s) = So_{21}(s) = e^{-Q_1(s)q\left({z_2}^{1/q}-{z_1}^{1/q}\right)}, \quad (9c)$$

where Ei(1,x) is an exponential integral and $q=2/(\alpha+\beta+2)$. Scattering parameters in the case of the Bessel line can be determined analytically. For comparative purposes, scattering parameters were calculated for the Bessel line for α = -1 and β=1. The exact parameter S_{12} for this case is :

$$S_{12}(s) = \frac{(FK_m(a) \cdot FI_m(a) - FI_p(a) \cdot FK_p(a)}{FI_m(a) \cdot FK_m(b) - FK_p(a) \cdot FI_p(b)},$$

$$FI_{p/m}(z) = (R_o^{-1} I_0(\gamma z) \pm Y_{co} I_1(\gamma z))\sqrt{z}R_o/2, \qquad (10)$$
$$FK_{p/m}(z) = (R_o^{-1} K_0(\gamma z) \pm Y_{co} K_1(\gamma z)\,)\sqrt{z}R_o/2,$$
$$\gamma = \sqrt{Z(s)Y(s)},\ Y_{co} = \sqrt{\frac{Z(s)}{Y(s)}}, z_1 = a,\ z_2 = b\ ,$$

where $I_n(z)$ and $K_n(z)$ are modified Bessel functions of the first and second kind respectively. The approximate scattering parameters for the Bessel line easily obtained from equations (9), where we need to substitute $\alpha = -1$, $\beta = 1$ and q=1.

The results for the first approximation of the So_{11}, So_{22} are very good for the most cases (see [6]) but for So_{12} we try to calculate the second order approximation to obtain a better fitting.

C. *Second order approximation of So_{12} for the Bessel transmission line*

The calculation of the second order S_{12} formula gives analytical solution of the form

$$S1_{12}(\omega) = S1_{21}(\omega) = e^{-qQ_1(\omega)C_{12}}\big(S1a_{12}(\omega) + S1b_{12}(\omega) + S1c_{12}(\omega)\big) \qquad (11)$$

Where:

$$S1a_{12}(\omega) = Q(\omega)\left(qQ_2(\omega)C_{12} + Q(\omega)\left(e^{-a(\omega)C_{12}} - 1\right)\right)$$

$$S1b_{12}(\omega) = B^2\left(G\left(-a(\omega)z_1^{\frac{1}{q}}\right) - G\left(-a(\omega)z_2^{\frac{1}{q}}\right) + Ei(1, -a(\omega)z_1^{\frac{1}{q}})\left(Ei(1, a(\omega)z_2^{\frac{1}{q}}) - Ei(1, a(\omega)z_1^{\frac{1}{q}})\right)\right)$$

$$S1c_{12}(\omega) = Q(\omega)B\left[e^{-a(\omega)z_2^{\frac{1}{q}}}Ei\left(1, -a(\omega)z_2^{\frac{1}{q}}\right) + e^{a(\omega)z_1^{\frac{1}{q}}}Ei\left(1, a(\omega)z_1^{\frac{1}{q}}\right)\right] + Ei(1, a(\omega)z_2^{\frac{1}{q}})e^{a(\omega)z_1^{\frac{1}{q}}} - Ei(1, a(\omega)z_1^{\frac{1}{q}})e^{-a(\omega)z_2^{\frac{1}{q}}}$$

and the function/constants used have the form:

$$C_{12} = z_2^{\frac{1}{q}} - z_1^{\frac{1}{q}}, Q(\omega) = \frac{Q_2(\omega)}{2Q_1(\omega)}, B = \frac{\alpha - \beta}{4}q$$

and G(x) is Meijer G function for specified arguments:

$$G(x) = G_{2\,3}^{3\,1}\left(x\middle|{}^{0,1}_{0,0,0}\right)$$

As one can see in the second order approximation occurs the coefficient equal to $So_{12}(s)$ multiplied by some coefficients $S1a_{12}(\omega, z_1, z_2)$, $S1b_{12}(\omega, z_1, z_2)$, $S1c_{12}(\omega, z_1, z_2)$. The estimations of the $S1_{12}(\omega, z_1, z_2)$ shows that only the first few samples of the second order approximation has important impact for the S_{12} parameter (see Fig. 1), than we use the second order approximation only for them. We do not use the whole second order approximation due to using the simplified formula for $Ei(1, x)$ appropriate for small arguments value.

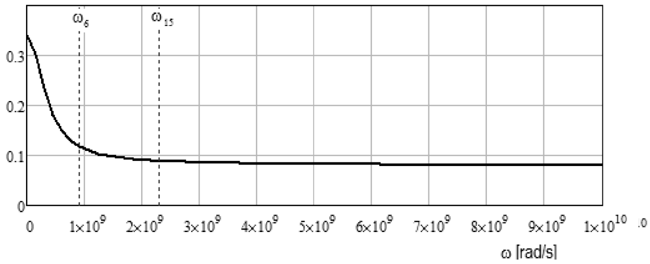

Fig.1 The estimated maximal values of the second order approximation.

The formula for the approximate $S_{12}(\omega)$ for the arbitrary frequency takes the following form:

$$S_{a12}(\omega) = So_{21}(\omega) + S1_{21}(\omega)\ for\ \omega \le \omega_q$$
$$S_{a12}(\omega) = So_{21}(\omega)\ for\ \omega > \omega_q \qquad (12)$$

where ω_q is as small as possible.

IV. Results

As an example we have considered a nonuniform (Bessel) interconnect with frequency dependent parameters shown in Fig.2.

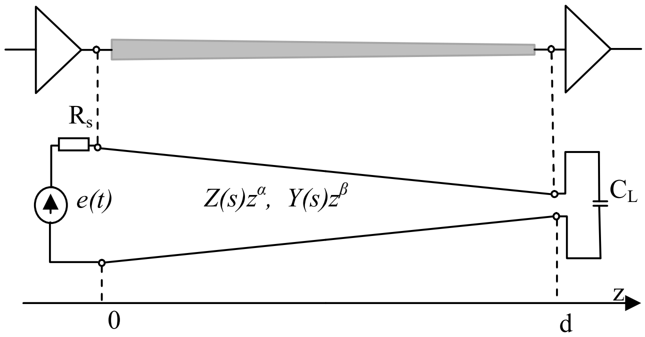

Fig.2 The considered system inverter-interconnect-inverter and its circuit model.

The longitudinal parameters of the interconnect Z(ω), Y(ω) depend on the frequency as follows:

$$Z(s) = R + sL + \left(0.1 + 10^{-4.5}\sqrt{s}\,\right) = Z_0(s) + Z_1(s),$$

$$Y(s) = G + sC\left(1 + \frac{\varepsilon_s/\varepsilon_\infty - 1}{1 + p\tau}\right) = Y_0(s) + Y_1(s),$$

where R=10Ω, L= 2nH, G=10μS, C=1pF, $\varepsilon_s = 4$, $\varepsilon_\infty = 1$, τ = 2ns, d=10cm.

The frequency dependence of the PUL parameters of interconnect R(ω), L(ω), G(ω), C(ω) are shown [6]. The above relationships of longitudinal parameters Z(ω),Y(ω) of the interconnect as functions of frequency were approximated by means of rational functions using very efficient algorithm-vector fitting [5]. The results for the first order approximation of So_{11} and So_{22} can be seen in [6] than we show there only the second order impact for the S_{12}. Frequency characteristics of the module $|S_{12}(\omega)|$ to compare the exact calculation from (10) with approximated formulas (9c) and (12) are shown in Fig.3. To calculate (12) we assume that we take into account first 15 samples in frequency domain, which gives ω_q =2.3e10[rad/s]

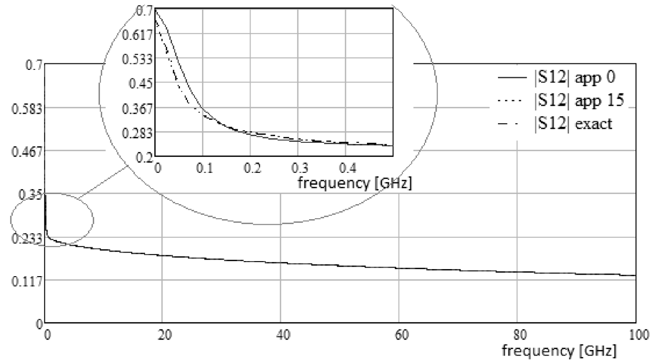

Fig.3 Dependence of scattering parameters: |S12| app 0 - first order approximation, |S12| app 15 - first order with 15 first samples of second order and |S12| exact -exact of nonuniform (Bessel) transmission line on frequency.

On the base of scattering parameters one can calculate the formulas for the voltage response. The next figures (Fig 4,5) show the results for the considered Bessel transmission line model. A circuit consists of voltage pulse source (of the trapezoid shape A=2V, T_r =T_f =500ps, T_{on}=2ns) with source resistance R_s=150Ω and transmission line loaded by capacitor C_L=1pF.

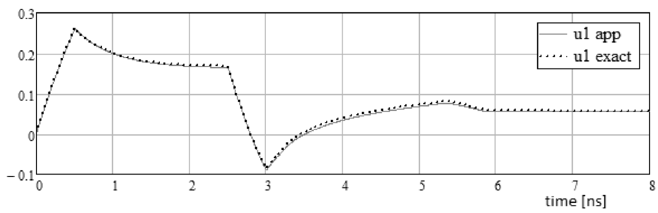

Fig.4 Near end voltages u1(t)-approximate and u1(t)-exact of the nonuniform (Bessel) transmission line.

The voltages at both ends of the Bessel transmission line were obtained based on the approximate (12) and exact scattering parameters in the frequency domain and they were subsequently transformed (IFFT) to the time domain. The near end voltages of the considered system (Fig.2) are shown in Fig.4. The differences visible in a steady state are small and can be explained by the fact, that during the simulation a finite number of terms (first term in our case) contribute to S_{11}, S_{12}, S_{21}, S_{22} were used.

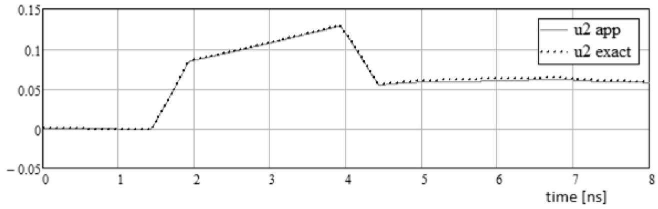

Fig.5 Far end voltages u2(t)-approximate and u2(t)-exact of the nonuniform (Bessel) transmission line.

The time of calculation the voltage for the presented approach due to the analytical form of scattering parameters is much shorter than calculation time of the voltage based on exact scattering parameters which can be calculated numerically.

The percentage error for the far end voltage is relatively small (see Fig 6). The error was calculated as:

$\delta = \frac{u2exact - u2app}{u2exa} 100\%$, where $u2exact$ is the far end voltage calculated using exact S_{12} (10), $u2app$ using S_{12} given in the article.

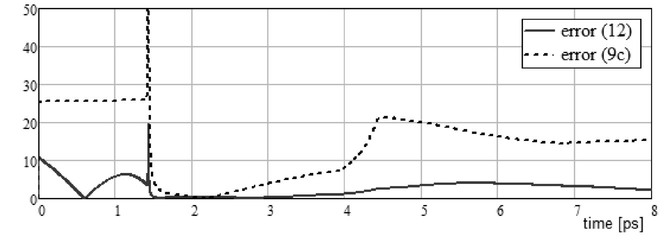

Fig.6 The error of the far end voltage calculated using S_{12} approximated. The solid line is approximation given by (9c) and the dashed one is calculated with (12).

V. CONCLUSIONS

We have shown that it is possible to generalize the approach based on the method of successive approximation for the case of a nonuniform transmission line with frequency dependent parameters. As a result, we obtain a closed form (meaning a first order approximation) of scattering parameters of nonuniform transmission line in frequency domain. The approximation is satisfactory for the considered transmission line. Equations (10) allow us to determine the approximate scattering parameters for nonuniform lines by integrating analytically (such as in the case of the Bessel lines) or numerically and applying the approximation by rational functions using a vector fitting algorithm. Compared with the approach based on dyadic Green's function and parametric macromodeling applied to weakly nonuniform transmission lines [2,3] the presented approach is simpler. The presented approach permits the implementation of the model in the SPICE program.

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